Equations (7, 10, and 11) can be solved to give the following expressions of θ and ψ up to and including terms of second power:

$$\begin{aligned} \theta &= \alpha_{31} + C_3(C_1 + C_4) \\ \psi &= \alpha_{32} + C_6(C_1 + C_4) \end{aligned} \tag{12}$$

Erroneous results of θ and ψ are given in Ref. 1.

Substituting Eqs. (6, 10, 11, and 12) into the first, second, and fourth of Eqs. (4), and making use of the approximation that the ratio z/R is small compared to unity, one has

$$\left.\begin{aligned} \epsilon_{11} &= \bar{\epsilon}_{11} + z\chi_{11} + z^2\nu_{11} \\ \epsilon_{22} &= \bar{\epsilon}_{22} + z\chi_{22} + z^2\nu_{22} \\ \epsilon_{12} &= \bar{\epsilon}_{12} + z\chi_{12} + z^2\nu_{12} \end{aligned}\right\} \tag{13}$$

where

$$\left.\begin{aligned}
\bar{\epsilon}_{11} &= C_1 + \tfrac{1}{2}(C_1^2 + C_2^2 + C_3^2) \\
\chi_{11} &= [(C_1/R_1) + D_1] + (1/R_1)(C_1^2 + C_2^2 + C_3^2) + (C_1D_1 + C_2D_2 + C_3D_3) \\
\nu_{11} &= \frac{1}{R_1}\left[\left(\frac{C_1}{R_1}\right) + D_1\right] + \frac{3}{2R_1^2}(C_1^2 + C_2^2 + C_3^2) + \left(\frac{2}{R_1}\right)(C_1D_1 + C_2D_2 + C_3D_3) + \tfrac{1}{2}(D_1^2 + D_2^2 + D_3^2) \\
\bar{\epsilon}_{22} &= C_4 + \tfrac{1}{2}(C_4^2 + C_5^2 + C_6^2) \\
\chi_{22} &= [(C_4/R_2) + D_4] + (1/R_2)(C_4^2 + C_5^2 + C_6^2) + (C_4D_4 + C_5D_5 + C_6D_6) \\
\nu_{22} &= \left(\frac{1}{R_2}\right)\left(\frac{C_4}{R_2} + D_4\right) + \left(\frac{3}{2R_2^2}\right) \times (C_4^2 + C_5^2 + C_6^2) + \left(\frac{2}{R_2}\right) \times (C_4D_4 + C_5D_5 + C_6D_6) + \tfrac{1}{2}(D_4^2 + D_5^2 + D_6^2) \\
\bar{\epsilon}_{12} &= (C_2 + C_5) + (C_1C_5 + C_2C_4 + C_3C_6) \\
\chi_{12} &= \left(\frac{C_2}{R_1} + \frac{C_5}{R_2}\right) + (D_2 + D_5) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)(C_1C_5 + C_2C_4 + C_3C_6) + (C_1D_5 + C_5D_1 + C_2D_4 + C_4D_2 + C_3D_6 + C_6D_3) \\
\nu_{12} &= \left(\frac{C_2}{R_1^2} + \frac{C_5}{R_2^2}\right) + \left(\frac{D_2}{R_1} + \frac{D_5}{R_2}\right) + \left(\frac{1}{R_1^2} + \frac{1}{R_1R_2} + \frac{1}{R_2^2}\right)(C_1C_5 + C_2C_4 + C_3C_6) + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)(C_1D_5 + C_5D_1 + C_2D_4 + C_4D_2 + C_3D_6 + C_6D_3) + (D_1D_5 + D_2D_4 + D_3D_6)
\end{aligned}\right\} \tag{14}$$

The expressions $D_1 \ldots D_6$ are obtained from $B_1 \ldots B_6$, respectively, by the transformation $u \to \theta$, $v \to \psi$, $w \to \chi$, i.e., $D_3 = [(1/A_1)(\partial\chi/\partial\alpha_1)] + \theta/R_1$.

Because of the error in deriving the expressions for θ, ψ, χ, the strain-displacement relations given in Eqs. (VI. 41) and (VI. 42) of Ref. 1 are in error and should be revised. For the same reason, the corresponding formulas for the flat plate in Sec. 46 of Ref. 1 should be revised.

When both X, Y, Z and α_1, α_2, z are right-hand coordinate systems, Eqs. (2) automatically give the correct signs of the principal radii of curvature R_1 and R_2. The formulas given here for the general shell can be readily transformed to obtain the formulas for special shells, such as spherical, conical, cylindrical, or toroidal shells, for flat plates, and for curved and straight beams.

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Prandtl Number Dependence of Heat Transfer in Falkner-Skan Flow

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SOME years ago, Spence[1] derived the Prandtl number (σ) dependence of the recovery and Reynolds-analogy factors for heat transfer to a flat plate with the density-viscosity product constant. He expanded the known temperature solution, first given by Pohlhausen, in a Taylor series in $(\sigma - 1)$, and expressed both factors as the product of 1) the value at $\sigma = 1$, 2) a power of σ, and 3) a series of the form $1 + K(\sigma - 1)^2 + \ldots$. The results confirmed the fitted $\sigma^{1/2}$ dependence of the recovery factor suggested by Pohlhausen and modified his suggested $\sigma^{-2/3}$ dependence of the Reynolds-analogy factor to $\sigma^{-0.64885}$. (It is interesting to note that $\sigma^{-0.65}$ had been suggested by Crocco[2] without any derivation.) The factor K, which indicates the accuracy of the power law, was also computed by Spence,[1] but given incorrectly in his paper.

The purpose of this note is to correct the values of K given by Spence, and to give similar results for the heat transfer to two-dimensional and axisymmetric stagnation points in incompressible flow.

Examination of Spence's equation (9) shows that the constant K given in Eq. (10) as $+ 0.0095$ should be -0.0095, so that the recovery factor r is

$$r = \sigma^{1/2}\,[1 - 0.0095\,(\sigma - 1)^2 + \ldots]$$

This is in better agreement with Pohlhausen's exact calculations, which show $\sigma^{1/2}$ to be higher than the exact values, not lower.

A more serious error occurs in Spence's Reynolds-analogy factor s, where Spence gives $K = -0.347$, which would indicate that the power law was not a very good approximation to s. Actually, this number contains two errors, one of which is incorrect addition below Eq. (13) where 0.69421 should be 1.01433. But this would lead to $K = -0.50716$, which is also wrong. The source of the second error is in the left side of the second equality in Eq. (13), where the factor 1 should be 2. When this correction is also applied, we find $K = -0.00716$, so that

$$s = \sigma^{-0.64885}\,[1 - 0.00716\,(\sigma - 1)^2 + \ldots]$$

Now the accuracy of the power law becomes clear.

Similar results can be obtained for the recovery factor and heat-transfer rate in other cases of Falkner-Skan flow, where the pressure gradient is not zero, though a somewhat different method must be used since the explicit solution of the energy equation is not known in general. However, since only expansions about $\sigma = 1$ are desired, only the solution for $\sigma = 1$ is needed, and this easily can be found numerically. Details of derivation in the general case are given by Kemp.[3]

The results for two particularly interesting cases will be given here, namely, the two-dimensional and axisymmetric stagnation points. Exact values for the former have been given by Squire,[4] for the latter by Sibulkin,[5] and both have suggested a $\sigma^{0.4}$ power law representation of the heat-transfer parameter. The corresponding expansions are (in the notation of Squire[4] and Sibulkin[5]), respectively,

$$\alpha_3 = 0.570\,\sigma^{0.3892}\,[1 - 0.0092\,(\sigma - 1)^2 + \ldots] \quad \text{(two-dimensional)}$$

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$$C/(2)^{1/2} = 0.539\ \sigma^{0.3776}\ [1 - 0.0075(\sigma - 1)^2 + \ldots] \quad \text{(three-dimensional)}$$

One can see why the $\sigma^{0.4}$ law is a good representation. The more accurate powers make a noticeable difference only for $\sigma \geq 2$.

The importance of these power laws is enhanced by the well-known fact that they hold quite well in compressible flow and even for heat transfer from dissociating air.

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Reply by Author to N. H. Kemp

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The author must accept N. H. Kemp's corrections on the two arithmetical points he mentions and is pleased to know that they improve the theoretical basis for the 0.65 power law, which Crocco had found to agree very closely indeed with his computed values of s. The remarkable thing is that, as seen in Fig. 5 of Crocco's paper (Ref. 2 of the preceding comment), this agreement extends certainly as far as $\sigma = 2$. The asymptotic value as $\sigma \to \infty$, also obtained in the paper, is, however, 1.02 $\sigma^{-2/3}$. The further use that Kemp has made of the technique is very interesting.

The author would also like to take this opportunity to correct two minor misprints in his note:

1) On the left-hand side of the equation above (9), the quantity operated upon is $(r/\sigma^{1/2})$.

2) The term on the extreme left of Eq. (12) should carry the suffix $\sigma = 1$.

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